Phase boom for an electromagnetic wave in a ferromagnet

1996 J. Phys. A: Math. Gen. 29 2805

(http://iopscience.iop.org/0305-4470/29/11/015)

J. Phys. A: Math. Gen. **29** (1996) 2805–2825. Printed in the UK

Phase boom for an electromagnetic wave in a ferromagnet

H Leblond

Institut de Recherche Mathématique de Rennes, URA-CNRS 305, Université de Rennes I, Campus de Beaulieu, 35042 Rennes Cedex, France

Received 22 September 1995, in final form 21 December 1995

Abstract. We study the propagation of an electromagnetic wave with a high intensity in a ferromagnetic medium, in a way analogous to the mathematical theory of nonlinear geometric optics. We find that the evolution of the modulation of a quasi-monochromatic plane wave is described by a nonlinear transport equation. In the simplest case, we retrieve a result described by other models: the main nonlinear effect is a phase modulation proportional to both the time and the square of the amplitude of the wave.

But the most interesting feature is that the rapidly oscillating wave generates slowly varying waves that belong to soliton propagation modes, and that the latter react on the former by a phase factor. Two regimes occur, depending whether the slowly varying waves travel slower or faster than the modulation of the rapidly oscillating wave. An analogue to the boom of a supersonic airplane can thus be observed in the phase modulation of the incident wave.

1. Introduction

Various studies have already been devoted to the nonlinear modulation of a quasi-monochromatic electromagnetic wave in a ferromagnetic medium [1–3]. These theoretical studies, that have been confirmed experimentally [4–6], are all developed in the framework of a weakly nonlinear approximation that leads to an asymptotic model governed by the nonlinear Schrödinger (NLS) equation, or by a perturbed version of it. This model is obtained by use of a multiscale expansion method [7, 8]. It is well known that the result of such an expansion depends on the chosen scales. Physically, the question is: what does a 'weak' nonlinearity mean? The present work studies a case where the nonlinearity is weak, but not so weak as in the quoted papers.

The multiscale expansion scheme that leads to the NLS equation consists of three main steps: the first is the dispersion relation, the second a transport equation, the third the nonlinear evolution equation (the NLS equation in most cases). In all the mentioned cases, the transport equation is linear, and expresses only the fact that the wave envelope propagates at the group velocity, without deformation at this order of the perturbation scheme.

Recent mathematical works have emphasized nonlinear transport equations, defining the so-called nonlinear geometrical optics approximation. Such equations were derived formally in [9] from a general nonlinear PDE. A general mathematical theory and a proof of the convergence of the asymptotic model have been developed in [10]. A mathematical study of a model describing the propagation of an intense laser pulse in a nonlinear medium [11], investigated both a nonlinear Schrödinger model and a nonlinear transport equation. The latter was obtained for sources at a higher power level than the former.

Precisely, in the physical frame studied in [11], we have the following features: let us call ε the perturbative parameter of the multiscale expansion. It represents the ratio of a

0305-4470/96/112805+21$19.50

length typical of the wave modulation to the wavelength. Then the nonlinear Schrödinger model is obtained when the electric field of the wave has an order of magnitude of ε times an electric field characteristic of the medium. The nonlinear transport equation of the nonlinear geometrical optics approximation is obtained when the ratio between these two fields is close to $\sqrt{\varepsilon}$.

Following these ideas, we investigate in this paper the nonlinear modulation of a quasi-monochromatic wave in a saturated ferromagnetic dielectric, with a wave-field intensity much larger than that considered in previous works [1–3]. A nonlinear transport equation, which describes a nonlinear geometrical optics regime is derived. As in [11], the equation can be explicitly solved. We can thus study some physical properties of the wave; only the phase is affected. In the simple case of a plane wave, propagating along the direction of the external field, the modulation of the phase is proportional to the square of the amplitude of the wave. The same model also describes the interaction between the quasi-monochromatic wave and slowly varying solitary waves. The fact that these latter waves can propagate in such a medium is already known. This interaction only occurs if the wave is modulated in the transverse direction.

The system that describes the interaction is derived in section 2, and reduced in order to make the structure of the solitary wave and of the interaction apparent in section 3. It may have two behaviours, depending on whether the rapidly oscillating wave travels slower or faster (group velocity) than the solitary waves. The discussion is achieved in section 4 in the particular case where the propagation is parallel to the external magnetic field. We give the solution of the interaction system for arbitrary initial data, and a physical interpretation for some special cases. It is found that the phase modulation of the wave that we study in section 5 may present a singularity analogous to the boom of a supersonic airplane for sound waves.

2. The model and the multiscale expansion

As in our previous papers [2, 3], we use a classical model [1, 12, 13] that describes electromagnetic wave propagation in an isotropic, infinite ferromagnetic medium, with a linear behaviour with regard to the electric field. This model neglects inhomogeneous exchange interaction and damping. For lower wave intensities, in the linear framework, many properties of the wave propagation can be described without taking into account these effects [12, 14]. The inhomogeneous exchange term is important mainly when the wavelength is no longer very large in regard to the interatomic distances or in thin films [15]. We consider here typically microwaves frequencies, for which it can be *a priori* neglected. The present author has studied the influence of damping on the nonlinear modulation that is described by the NLS model, still in a ferromagnetic medium [16]. We show that the inhomogeneous exchange interaction, as soon as it can be treated as a perturbation, has no effect on the nonlinear modulation, at the scales where the formation of NLS solitons occurs. The appearance of a nonlinear effect of the inhomogeneous exchange for higher intensities seems unlikely. However, interactions between the microwave frequency under consideration and spin waves might occur. The effect of such interactions on the ferromagnetic resonance absorption is studied in [17]. This kind of treatment cannot be incorporated in the frame of this paper. Fortunately, these effects can be avoided experimentally [4].

Under these assumptions, the magnetization density $\boldsymbol{M}$ and the magnetic field $\boldsymbol{H}$ must satisfy the equations:

$$-\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{H})+\Delta\boldsymbol{H}=\frac{1}{c^2}\partial_t^2(\boldsymbol{H}+\boldsymbol{M}) \tag{1}$$

$$\partial_t\boldsymbol{M}=-\mu_0\hat{\delta}\boldsymbol{M}\wedge\boldsymbol{H} \tag{2}$$

where $c=1/\sqrt{\hat{\varepsilon}\mu_0}$ is the speed of light based on the dielectric constant $\hat{\varepsilon}$ of the medium, μ_0 the magnetic permeability in vacuum and $\hat{\delta}$ the gyromagnetic ratio ($\partial_t u$ denotes the partial derivative of the function u with respect to the variable t). We first rescale t, $\boldsymbol{H}$, $\boldsymbol{M}$ into ct, $(\hat{\delta}\mu_0/c)\boldsymbol{H}$, $(\hat{\delta}\mu_0/c)\boldsymbol{M}$, and write the system (1), (2) in the form:

$$-\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{H})+\Delta\boldsymbol{H}=\partial_t^2(\boldsymbol{H}+\boldsymbol{M}) \tag{3}$$

$$\partial_t\boldsymbol{M}=-\boldsymbol{M}\wedge\boldsymbol{H}. \tag{4}$$

Now we introduce a multiscale expansion corresponding to the following assumptions:

(i) The medium is magnetized to saturation by an external magnetic field.

(ii) A quasi-monochromatic wave propagates along the x-axis with a slowly varying envelope.

(iii) The amplitude of this wave is small in regard to the exterior field, but large compared to the amplitude of the wave studied in [3].

Thus $\boldsymbol{M}$ is first expanded in a series of harmonics of the fundamental $\mathrm{e}^{\mathrm{i}\phi}$, with

$$\phi=kx-\omega t \tag{5}$$

$$\boldsymbol{M}=\sum_{n\in\mathbb{Z}}\boldsymbol{M}^n\mathrm{e}^{\mathrm{i}n\phi} \tag{6}$$

and we have the reality condition $\boldsymbol{M}^{n*}=\boldsymbol{M}^{-n}$ (asterisk denotes complex conjugation). Then each amplitude $\boldsymbol{M}^n$ is expanded in a power series of a small parameter ε, that measures the ratio between the saturation magnetization and the amplitude of the wave:

$$\boldsymbol{M}^n=\boldsymbol{M}_1^n+\varepsilon\boldsymbol{M}_0^n+\varepsilon^2\boldsymbol{M}_2^n+\cdots. \tag{7}$$

$\boldsymbol{H}$ is expanded in the same way as $\boldsymbol{M}$, and each $\boldsymbol{M}_k^n$, $\boldsymbol{H}_k^n$ is assumed to be a function of the slow variables $\boldsymbol{\xi}=(\xi,\eta,\zeta)$ and τ defined by

$$\partial_t=\varepsilon^2\partial_\tau \tag{8}$$

$$\boldsymbol{\nabla}_{\boldsymbol{x}}=\varepsilon^2\boldsymbol{\nabla}_{\boldsymbol{\xi}}. \tag{9}$$

The term of order zero is assumed to be uniform and constant, and represents the field created inside the sample by the external field. We will call it the exterior field, although demagnetizing factors should be taken into account. It reads:

$$\boldsymbol{M}_0^0=\boldsymbol{m}\qquad\boldsymbol{H}_0^0=\alpha\boldsymbol{m}. \tag{10}$$

The y- and z-axes are chosen so that

$$\boldsymbol{m}=\begin{pmatrix}m_x\\m_t\\0\end{pmatrix}=\begin{pmatrix}m\cos\varphi\\m\sin\varphi\\0\end{pmatrix}. \tag{11}$$

The particular case $\varphi=0$, where the propagation direction is parallel to the exterior field, is much simpler than the general case. Most quantities can be computed explicitly. We will take advantage of this feature for going further in the discussion, in this particular case. It will be referred to as the longitudinal case thereafter.

Among the terms of order ε, only the coefficients $\boldsymbol{M}_1^1$ and $\boldsymbol{H}_1^1$ of the fundamental (and their complex conjugate) will be non-zero, corresponding to a quasi-monochromatic wave.

All other terms are assumed to vanish at infinity. Thus, the parameter ε is such that the typical length L of the studied phenomenon has an order of magnitude of $1/\varepsilon^2$ times the wavelength λ. The ratio of the amplitude of the wave divided by the magnitude of the exterior field is proportional to $\varepsilon = \sqrt{\lambda/L}$, instead of λ/L in the case studied in [1, 3].

The order-by-order resolution of the perturbative scheme is described in appendix 1. The terms of order ε^1 of the fundamental are given by

$$\boldsymbol{H}_1^1 = \boldsymbol{h}_1^1 g(\boldsymbol{\xi}, \tau) \tag{12}$$

$$\boldsymbol{M}_1^1 = \boldsymbol{m}_1^1 g(\boldsymbol{\xi}, \tau) \tag{13}$$

g is an arbitrary function of the stretched variables $(\boldsymbol{\xi}, \tau)$ defined by (8), (9), and $\boldsymbol{h}_1^1$ and $\boldsymbol{m}_1^1$ are constant polarization vectors, given in appendix 1 (equations (94) and (95)). We find also that ω and k must satisfy the dispersion relation (studied in [13, 3]):

$$\mu^2 m_x^2 + \gamma\mu(1+\alpha)m_t^2 = \gamma^2\omega^2 \tag{14}$$

where we have put

$$\gamma = 1 - \frac{k^2}{\omega^2} \qquad \text{and } \mu = 1 + \alpha\gamma. \tag{15}$$

The solvability condition of equation (4) at order ε^3 gives the following equation:

$$\mathrm{i}A\partial_\tau g + \mathrm{i}\mathcal{D}g + B_0 g|g|^2 + F(\boldsymbol{H}_2^0, \boldsymbol{M}_2^0)g = 0. \tag{16}$$

A and B_0 are real constants, $\mathcal{D}g$ is a first-order spatial partial derivative of g, and $F(\boldsymbol{H}_2^0, \boldsymbol{M}_2^0)$ is a linear function of the fields $\boldsymbol{H}_2^0$ and $\boldsymbol{M}_2^0$ (equations (111) to (114)). Equation (16) is a nonlinear transport equation for the amplitude g. It contains the self-interaction term $B_0 g|g|^2$, but also the term $F(\boldsymbol{H}_2^0, \boldsymbol{M}_2^0)g$. This latter term corresponds to an interaction between the rapid oscillating wave and the long solitary waves, which can be described by the quantities $\boldsymbol{H}_2^0$ and $\boldsymbol{M}_2^0$. The main difficulty of the present work is to describe these solitary waves correctly.

In order to find their evolution equations (the equations that relate the fields $\boldsymbol{M}_2^0$ to $\boldsymbol{H}_2^0$), we consider the conditions deduced from equations (3). They are trivial at order ε^2, thus the sought equations are found at order ε^6. They are:

$$\partial_\tau^2(H_2^{0,x} + M_2^{0,x}) = -\partial_\xi(\partial_\eta H_2^{0,y} + \partial_\zeta H_2^{0,z}) + (\partial_\eta^2 + \partial_\zeta^2)H_2^{0,x} \tag{17}$$

and the relations deduced from (17) by circular permutation of the axes (x, y, z). We use the notation:

$$\begin{aligned} H_2^{0,x} &= \Phi \\ H_2^{0,y} &= \Psi \\ H_2^{0,z} &= \Xi \end{aligned} \tag{18}$$

and, solving the perturbative scheme up to order ε^4, we find conditions that enable us to eliminate the field $\boldsymbol{M}_2^0$ from expression (114) of $F(\boldsymbol{H}_2^0, \boldsymbol{M}_2^0)$ in equation (16), and from equations (17). Finally we get the system:

$$\mathrm{i}A\partial_\tau g + \mathrm{i}\mathcal{D}g + Bg|g|^2 + Cg\Phi + Dg\Psi + Eg\int_{-\infty}^{\tau}\mathcal{D}|g|^2 = 0 \tag{19}$$

$$\begin{aligned} \left(\partial_\eta^2 + \partial_\zeta^2 - \frac{\alpha + \sin^2\varphi}{\alpha}\partial_\tau^2\right)\Phi = {} & \frac{-\sin\varphi\cos\varphi}{\alpha}\partial_\tau^2\Psi + \partial_\xi(\partial_\eta\Psi + \partial_\zeta\Xi) + a\partial_\tau^2|g|^2 \\ & + \frac{\gamma m_t}{m^2}\partial_\tau\mathcal{D}|g|^2 \end{aligned} \tag{20}$$

$$\left(\partial_\xi^2+\partial_\zeta^2-\frac{\alpha+\cos^2\varphi}{\alpha}\partial_\tau^2\right)\Psi=\frac{-\sin\varphi\cos\varphi}{\alpha}\partial_\tau^2\Phi+\partial_\eta(\partial_\xi\Phi+\partial_\zeta\Xi)+b\partial_\tau^2|g|^2+\frac{\gamma m_t}{m^2}\partial_\tau\mathcal{D}|g|^2 \tag{21}$$

$$\left(\partial_\xi^2+\partial_\eta^2-\frac{\alpha+1}{\alpha}\partial_\tau^2\right)\Xi=\partial_\zeta(\partial_\xi\Phi+\partial_\eta\Psi) \tag{22}$$

(the constants are given in appendix 1). We have put the nonlinear transport equation (16) for g into the more convenient form (19). Equations (20)–(22) describe the evolution of long solitary waves in the medium, supported by the quantity $\boldsymbol{H}_0^2=(\Phi,\Psi,\Xi)$, and the generation of such waves by the rapidly oscillating wave with amplitude $|g|$. In the next section, we will reduce these equations, to make the various interacting modes appear.

3. The solution of the interaction system

Equation (19) can be solved by separating the phase and the amplitude of g (see appendix 2). The solution reads:

$$g=r\mathrm{e}^{\mathrm{i}\theta} \tag{23}$$

where r and θ are given by

$$r=r(\boldsymbol{\xi}')=r(\xi',\eta',\zeta') \tag{24}$$

$$\theta(\boldsymbol{\xi}',\tau')=\Lambda r^2(\boldsymbol{\xi}')\tau'+\frac{C}{A}\int_{\tau_1}^{\tau'}\Phi(\boldsymbol{\xi}',\hat\tau)\,\mathrm{d}\hat\tau+\frac{D}{A}\int_{\tau_2}^{\tau'}\Psi(\boldsymbol{\xi}',\hat\tau)\,\mathrm{d}\hat\tau. \tag{25}$$

(The ‘integration constants’ τ_1 and τ_2 are *a priori* arbitrary functions of $\boldsymbol{\xi}'$, and $\Lambda=B/A-E$.) We use the change of variables:

$$\begin{aligned}\xi'&=\xi-v_\xi\tau\\ \eta'&=\eta-v_\eta\tau\\ \zeta'&=\zeta\\ \tau'&=\tau\end{aligned} \tag{26}$$

with

$$v_\xi=\frac{(b+1)u}{b+1+\gamma\mu u^2} \tag{27}$$

$$v_\eta=\frac{\gamma\mu^2 m_x m_t u}{\Gamma(b+1+\gamma\mu u^2)}. \tag{28}$$

We have put

$$u=\frac{\omega}{k}\qquad\Gamma=\gamma^2\omega^2\qquad b=\frac{\mu^2 m_x^2}{\gamma^2\omega^2}. \tag{29}$$

Thus r is a constant in the frame moving at the velocity $(v_\xi,v_\eta,0)$.

Differentiating the dispersion relation (14), we can compute the group velocity $\mathrm{d}\omega/\mathrm{d}k$ of the wave, in the x direction. We find that

$$\frac{\mathrm{d}\omega}{\mathrm{d}k}=v_\xi. \tag{30}$$

In the same way, we can compute the dispersion relation for the phase $\tilde{\phi} = (kx + ly + pz - \omega t)$, with an arbitrary propagation direction. Differentiating, then setting l and m equal to zero, we find that

$$\left.\frac{\partial\omega}{\partial l}\right|_{l=0,\,p=0} = v_\eta \qquad \text{and} \qquad \left.\frac{\partial\omega}{\partial p}\right|_{l=0,\,p=0} = 0. \tag{31}$$

Thus the group velocity of the wave under consideration has a non-zero transverse component v_η, in the y direction. Because of the external field, the medium is no longer isotropic; that is why the group velocity is not parallel to the phase velocity. The amplitude propagates at the group velocity, without any deformation due to nonlinearity. As in [11], the nonlinearity affects only the phase θ.

Let us consider equation (25). A phase shift proportional to both the intensity $r^2(\boldsymbol{\xi}')$ of the wave and the time τ' appears, as in [11].

This is a well-known effect, that is also described by the NLS model. In [3], we obtained the equation:

$$\mathrm{i}A_0\partial_\tau g + B_0\partial_\xi^2 g + C_0 g|g|^2 + D_0\lambda g = 0 \tag{32}$$

with:

$$\lambda = \lim_{\xi\to-\infty} |g|^2. \tag{33}$$

Equation (32) describes the same quantity g as in the present paper, but, as we have already noted, with other space, time and amplitude scales. Its solution reads:

$$g(\xi,\tau) = \psi(\xi,\tau)\exp\left(\mathrm{i}\frac{D_0}{A_0}\lambda\tau\right) \tag{34}$$

where ψ is a solution of the NLS equation obtained by setting $D_0 = 0$ in equation (32). The phase factor $(D_0/A_0)\lambda\tau$ in equation (34) is analogous to the factor $\Lambda r^2(\boldsymbol{\xi}')\tau'$ in equation (25). The quantity $r^2(\boldsymbol{\xi}')$, which appears in equation (25), is the square amplitude at the considered point, while λ, which appears in equation (32), is the same quantity at the infinity of space. This difference is justified by the scalings. Indeed, if we call ε the ratio of the amplitude of the wave to the external field in both the NLS model and the present one, equation (32) is obtained at a space scale of order $1/\varepsilon$, and equation (25) at a space scale of order $1/\varepsilon^2$, which is infinity in comparison with the former. In fact, the phase factor $(D_0/A_0)\lambda\tau$ does not coincide exactly with the factor $\Lambda r^2(\boldsymbol{\xi}')\tau'$, but rather, at least in the longitudinal case, with the term $\Lambda_1 r^2(\boldsymbol{\xi}')\tau'$ of equations (82) and (83). Such a phase factor has been computed in another way in our study of the nonlinear Faraday effect in the same medium [18]. In the longitudinal case, the present factor coincides with the phase factor proportional to ρ_1^2 in equations (29*ab*) of [18] (we take the limit of Λ as $\omega \to +\infty$, and take into account the fact that $g_{1,2} = m_x g$, according to equation (27*a*) of [18]).

The unexpected feature in expression (25) of the modulation of the wave is the other phase factors that describe an action of the terms Φ and Ψ on the phase. Such an interaction does not appear in the usual nonlinear geometrical optics [11]. The difference is partly due to the fact that we do not use exactly the same expansion as it was used there (note that this implies that the mathematical result of [10] does not apply to our work). We think that the present expansion should not lead to a closed system in the case of the laser propagation model studied in [11], and thus that no interaction analogous to the present one should appear there.

We will see that the quantities Φ and Ψ can describe long solitary waves. The equations that give their evolution are reduced first by use of a rotation around the z (or ζ) axis. We use the coordinates (X, Y) such that the X-axis lines up with the exterior field. (Φ_1, Φ_2, Ξ) are the components of the field $\boldsymbol{H}_2^0$ in the rotated frame. Second the system is decoupled into three equations by using the following transform:

$$\begin{aligned} \Psi_1 &= \partial_Y \Phi_2 + \partial_\zeta \Xi \\ \Psi_2 &= \partial_\zeta \Phi_2 - \partial_Y \Xi \end{aligned} \tag{35}$$

Φ_2 and Ξ can be recovered from Ψ_1 and Ψ_2 by solving the Poisson equations:

$$\begin{aligned} (\partial_Y^2 + \partial_\zeta^2)\Phi_2 &= \partial_Y \Psi_1 + \partial_\zeta \Psi_2 \\ (\partial_Y^2 + \partial_\zeta^2)\Xi &= \partial_\zeta \Psi_1 - \partial_Y \Psi_2. \end{aligned} \tag{36}$$

(Φ_2 and Ξ are assumed to vanish at infinity.) The equations that govern the evolution of Ψ_1, Ψ_2, Φ_1 are linear, and the fields can thus be written as the sum of terms corresponding to 'free waves' and terms describing a wave that accompanies the modulation of the high frequency. We introduce functions χ_1 and χ_2 (defined by equations (150) and (151) in appendix 2). They verify the equations:

$$(V_0^2 \partial_X^2 + \partial_Y^2 + \partial_\zeta^2 - \partial_\tau^2)\chi_1 = 0 \tag{37}$$

$$(V_0^2(\partial_X^2 + \partial_Y^2 + \partial_\zeta^2) - \partial_\tau^2)\chi_2 = 0. \tag{38}$$

We have put

$$V_0 = \sqrt{\frac{\alpha}{1+\alpha}}. \tag{39}$$

Equation (38) corresponds to a wave propagation at velocity V_0. It is well known that V_0 is a propagation velocity for solitary waves in this medium [19, 20]. Equation (37) also corresponds to a wave propagation, but with an anisotropic velocity. The velocity is 1 (in our units, it is the light velocity based on the dielectric constant of the medium) in the Y and ζ directions, and V_0 along the X direction, which is the direction of the exterior field.

Now we can give an explicit expression for the part of the field $\boldsymbol{H}_2^0$ that corresponds to this homogeneous solution. In the (X, Y, ζ) frame:

$$\boldsymbol{H}_2^0 = \begin{pmatrix} 0 \\ \partial_\zeta \\ -\partial_Y \end{pmatrix} \chi_2 + \begin{pmatrix} \partial_Y^2 + \partial_\zeta^2 \\ -V_0^2 \partial_X \partial_Y \\ -V_0^2 \partial_X \partial_\zeta \end{pmatrix} \chi_1 + \boldsymbol{H}_2^{0,+}. \tag{40}$$

Here $\boldsymbol{H}_2^{0,+}$ is a particular solution (Φ^+, Ψ^+, Ξ^+) of the complete system (with a non-zero value of r). In an analogous way, we obtain

$$\boldsymbol{M}_2^0 = \frac{1}{\alpha} \begin{pmatrix} 0 \\ \partial_\zeta \\ -\partial_Y \end{pmatrix} \chi_2 - \frac{1}{1+\alpha} \begin{pmatrix} 0 \\ \partial_Y \\ \partial_\zeta \end{pmatrix} \partial_X \chi_1 + \boldsymbol{M}_2^{0,+} \tag{41}$$

$\boldsymbol{M}_2^{0,+}$ is obviously the particular solution of the complete system that corresponds to $\boldsymbol{H}_2^{0,+}$. Let us now seek solutions to equations (37), (38) in the form:

$$\chi_j(X, Y, \zeta, \tau) = \chi_j^0(X \cos\varphi_0 + Y \sin\varphi_0 - V_j \tau). \tag{42}$$

Such a solution describes a plane wave propagating in the direction that makes an angle φ_0 with the X-axis in the (XY) plane.

For χ_1, we find that

$$V_1 = \sqrt{\frac{\alpha + \sin^2\varphi_0}{1+\alpha}} \tag{43}$$

and the corresponding component of $\boldsymbol{H}_2^0$ reads:

$$\boldsymbol{H}^0_{2,(\chi_1)} = \begin{pmatrix} (1+\alpha)\sin\varphi_0 \\ -\alpha\cos\varphi_0 \\ 0 \end{pmatrix} \frac{\sin\varphi_0}{1+\alpha}\chi_1^{0''}(X\cos\varphi_0 + Y\sin\varphi_0 - V_1\tau). \tag{44}$$

$\boldsymbol{H}^0_{2,(\chi_1)}$ stays in the plane defined by the external field and the propagation direction; for very large values of α, it is transverse to the propagation direction. For χ_2, we find that $V_2 = V_0$, and the corresponding component of $\boldsymbol{H}_2^0$ reads:

$$\boldsymbol{H}^0_{2,(\chi_2)} = \begin{pmatrix} 0 \\ 0 \\ -\sin\varphi_0 \end{pmatrix} \chi_2^{0'}(X\cos\varphi_0 + Y\sin\varphi_0 - V_0\tau) \tag{45}$$

$\boldsymbol{H}^0_{2,(\chi_2)}$ is perpendicular to both the external field and the propagation direction. Recalling that $\boldsymbol{H}_2^0$ is a quantity of order ε^2, it may be considered to be an infinitesimal variation of the field $\boldsymbol{H}_0^0$ in a precession move of this vector around the propagation direction. All these features allow us to recognize two propagation modes of long solitary waves, whose nonlinear behaviour has been studied in [19] and [20]. χ_1 corresponds to what is called in [20] the KdV mode, and χ_2 to Nakata's mode.

Let us now deal with the particular solution of the complete equations. In the frame moving at the group velocity $(v_\xi, v_\eta, 0)$, the square amplitude r^2 of the wave depends only on the spatial coordinates $\boldsymbol{\xi}' = (\xi', \eta', \zeta')$. Thus we search a particular solution Φ_1^+ of the equation that governs the evolution of Φ_1 (appendix 2, equation (146)), function of the variable $\boldsymbol{\xi}'$ only. Φ_1^+ must verify:

$$[(V_0^2 - v_X^2)\partial_X^2 - 2v_X v_Y \partial_X\partial_Y + (1 - v_Y^2)\partial_Y^2 + \partial_\zeta^2]\Phi_1^+ = \mathcal{R} \tag{46}$$

$\mathcal{R}$ is some functional of r^2, given in appendix 2 (equation (158)), and (v_X, v_Y) are the components of the group velocity in the (X, Y) frame. In order to reduce equation (46), let us consider the trinome $\mathcal{P}(U) = (V_0^2 - v_X^2)U^2 - 2v_X v_Y U + (1 - v_Y^2)$. $\mathcal{P}(U)$ is the difference of two squares when the reduced discriminant

$$\Delta' = v_X^2 - V_0^2 + V_0^2 v_Y^2 \tag{47}$$

is positive. When Δ' is negative, $\mathcal{P}(U)$ is the sum of two squares, affected by the sign of the quantity $(V_0^2 - v_X^2)$. But if $\Delta' < 0$, then $(V_0^2 - v_X^2) > 0$. Thus only two cases may occur:

If $\Delta' < 0$, then equation (46) can be reduced, by means of a linear change of coordinates $(X, Y) \mapsto (X_1, X_2)$, to the Poisson equation:

$$(\partial_\zeta^2 + \partial_{X_1}^2 + \partial_{X_2}^2)\Phi_1^+ = \mathcal{R}. \tag{48}$$

If $\Delta' > 0$, then equation (46) can be reduced, in the same manner, to the wave equation:

$$(\partial_\zeta^2 + \partial_{X_1}^2 - \partial_{X_2}^2)\Phi_1^+ = \mathcal{R}. \tag{49}$$

In the longitudinal case, $v_X = v$ and $v_Y = 0$, thus

$$\Delta' = v^2 - V_0^2 \tag{50}$$

and the condition on the sign of Δ' is easy to interpret (see below). In the general case, this condition ($v > V_0$ or $v < V_0$) is distorted by the existence of the transverse component of the group velocity v_Y. Unfortunately, the explicit expression of Δ' is very complicated and the discussion cannot be achieved algebraically in the general case.

4. The longitudinal case

We study in this section the special case where the exterior field is parallel to the propagation direction. Equations (19)–(22) reduce to

$$\mathrm{i}A\partial_\tau g + \mathrm{i}K\partial_\xi g + Bg|g|^2 + Cg\Phi + EKg\int_{-\infty}^{\tau}\partial_\xi|g|^2 = 0 \tag{51}$$

$$(\partial_\eta^2 + \partial_\zeta^2 - \partial_\tau^2)\Phi = \partial_\xi(\partial_\eta\Psi + \partial_\zeta\Xi) + a\partial_\tau^2|g|^2 - \frac{K}{m(\alpha+\delta\nu)}\partial_\tau\partial_\xi|g|^2 \tag{52}$$

$$\left(\partial_\xi^2 + \partial_\zeta^2 - \frac{1}{V_0^2}\partial_\tau^2\right)\Psi = \partial_\eta(\partial_\xi\phi + \partial_\zeta\Xi) \tag{53}$$

$$\left(\partial_\xi^2 + \partial_\eta^2 - \frac{1}{V_0^2}\partial_\tau^2\right)\Xi = \partial_\zeta(\partial_\xi\phi + \partial_\eta\Psi). \tag{54}$$

(The coefficients are given in appendix 3, equations (160)–(164).)

The reduction of the system is analogous to the general case, but much simpler. Equation (24) is still valid, with the change of coordinates:

$$\begin{aligned}\xi' &= \xi - v\tau\\ \eta' &= \eta\\ \zeta' &= \zeta\\ \tau' &= \tau.\end{aligned} \tag{55}$$

Owing to the symmetry of rotation of the system around the x-axis, the group velocity is now parallel to the propagation direction, and reads:

$$v = \frac{2(\alpha+\delta\nu)^{3/2}(\alpha+\delta\nu+1)^{1/2}}{2(\alpha+\delta\nu)(\alpha+\delta\nu+1)-\delta\nu}. \tag{56}$$

($\nu = \omega/m$ and $\delta = \pm 1$.) Equation (25) becomes

$$\theta(\boldsymbol{\xi}', \tau') = \Lambda r^2(\boldsymbol{\xi}')\tau' + \frac{C}{A}\int_{\tau_1}^{\tau'}\Phi(\boldsymbol{\xi}', \hat{\tau})\,\mathrm{d}\hat{\tau}. \tag{57}$$

The term that depends on Ψ disappears. The equation that describes the evolution of $\Phi = \Phi_1$ (analogous to equation (146) in appendix 2) is obtained. It reads:

$$(V_0^2\partial_\xi^2 + \partial_\eta^2 + \partial_\zeta^2 - \partial_\tau^2)\Phi = P\partial_\tau^2 r^2. \tag{58}$$

(The constants are given in appendix 3, equations (165), (166), (170).)

Let us now seek for a particular solution Φ^+ of equation (58). Owing to the τ dependency of r, we can choose Φ^+ as a function of $\boldsymbol{\xi}'$ only and equation (58) becomes

$$((V_0^2 - v^2)\partial_{\xi'}^2 + \partial_{\eta'}^2 + \partial_{\zeta'}^2)\Phi^+ = Pv^2\partial_{\xi'}^2 r^2. \tag{59}$$

Equation (59) is either of the form (48) or (49) depending on the sign of the quantity $(V_0^2 - v^2)$.

This condition has a very simple physical interpretation: the rapidly oscillating wave with amplitude r emits slowly varying waves of the mode described by Φ (which we call the KdV mode). These slowly varying waves propagate at their own velocity V_0. There are two possibilities then: the amplitude r of the rapidly oscillating wave can propagate either faster or slower than V_0. If it propagates slower ($v < V_0$), then the wave is emitted in all directions: this case is described by a Poisson equation in the frame moving with the fast oscillating wave. If it propagates faster ($v > V_0$), it outruns the emitted slowly varying wave, and if the input wave is localized, the emitted wave will concentrate on a cone, like the boom of an airplane with a supersonic speed. This latter case is described by equation (59) when it is a wave equation.

The sign of $(V_0^2 - v^2)$ can be explicitly determined here. For a wave with positive helicity ($\delta = +1$ with the notation of appendix 3), $V_0^2 - v^2 < 0$ and equation (59) is a wave equation, and for a wave with negative helicity ($\delta = -1$), two behaviours may appear: for $\omega > \omega_0$, where ω_0 is some threshold frequency, $V_0^2 - v^2 < 0$, and equation (59) is also a wave equation, but for $\omega < \omega_0$, $V_0^2 - v^2 > 0$, and equation (59) is a Poisson equation. An asymptotic value of ω_0 can be computed for strong exterior fields (α tends to $+\infty$):

$$\omega_0 = m\left(2\alpha + \frac{1}{2} + \frac{1}{8\alpha} + O\left(\frac{1}{\alpha^2}\right)\right). \tag{60}$$

In both cases $(V_0^2 - v^2) > 0$ or < 0, equation (59) can be solved explicitly by means of quadratures. Let

$$d = \sqrt{|V_0^2 - v^2|} \tag{61}$$

and

$$\xi_1 = \frac{\xi}{d} \qquad \boldsymbol{\xi}_1 = (\xi_1, \eta, \zeta). \tag{62}$$

Consider first the case $(V_0^2 - v^2) > 0$. Equation (59) writes

$$(\partial_{\xi_1}^2 + \partial_\eta^2 + \partial_\zeta^2)\Phi^+ = \rho(\xi_1, \eta, \zeta) \tag{63}$$

with

$$\rho = P v^2 \partial_\xi^2 r^2. \tag{64}$$

Solution of equation (63) is well known, it reads

$$\Phi^+(\boldsymbol{\xi}_1) = \int\!\!\int\!\!\int_{\mathbb{R}^3} \frac{\rho(\boldsymbol{u})\mathrm{d}\boldsymbol{u}}{\|\boldsymbol{\xi}_1 - \boldsymbol{u}\|}. \tag{65}$$

Formulae (64), (65) solve the equation for any source function r^2. Let us consider the following particular case, for which the expression of Φ is especially simple, and that has an interesting physical meaning:

$$r^2 = aH(\xi')\delta(\eta)\delta(\zeta) \tag{66}$$

a is a positive constant, $H(\xi')$ is the Heaviside function ($H(\xi') = 0$ if $\xi' < 0$, $H(\xi') = 1$ if $\xi' > 0$), and $\delta(\eta)$, $\delta(\zeta)$ are Dirac's δ-distributions relative to the variables η and ζ. Expression (66) represents the top front of a long pulse, with a very small transverse extension (or considered from a distance very large in regard to its transverse dimensions).

Then

$$\Phi^+ = \frac{aPv^2\xi'}{[\xi'^2 + d^2(\eta^2 + \zeta^2)]^{3/2}} \tag{67}$$

Φ^+ has an expression analogous to the expression of the electrostatic potential created by an electrostatic dipole put at the origin. The unit length in the transverse plane is modified by the coefficient $1/d$. Φ^+ is proportional to the maximal value a of the intensity r^2 of the rapidly oscillating wave, as expected.

In the case where $(V_0^2 - v^2) < 0$, equation (59) can be written as

$$(-\partial_{\xi_1}^2 + \partial_\eta^2 + \partial_\zeta^2)\Phi^+ = \rho(\xi_1, \eta, \zeta). \tag{68}$$

This is the wave equation in $2+1$ dimensions. Its properties are rather different from those of the wave equation in $3+1$ or $1+1$ dimensions, thus we find useful to recall its resolution. We define the Fourier transform in the (η, ζ) variables by

$$f(\xi_1, \eta, \zeta) = \int\int \tilde{f}(\xi_1, l, p) \exp(2\mathrm{i}\pi(l\eta + p\zeta))\, \mathrm{d}l\, \mathrm{d}p \tag{69}$$

$$\tilde{f}(\xi_1, l, p) = \int\int f(\xi_1, \eta, \zeta) \exp(-2\mathrm{i}\pi(l\eta + p\zeta))\, \mathrm{d}\eta\, \mathrm{d}\zeta. \tag{70}$$

Equation (68) gives

$$\partial^2_{\xi_1} \tilde{\Phi}^+ = -4\pi^2(l^2 + p^2)\tilde{\Phi} - \tilde{\rho} \tag{71}$$

which is easily solved:

$$\tilde{\Phi}^+ = \frac{\mathrm{i}}{2\kappa}\left[\mathrm{e}^{\mathrm{i}\kappa\xi_1} \int_{\xi_1^a}^{\xi_1} \mathrm{d}\hat{\xi}_1 \mathrm{e}^{-\mathrm{i}\kappa\hat{\xi}_1} \tilde{\rho}(\hat{\xi}_1, l, p) - \mathrm{e}^{-\mathrm{i}\kappa\xi_1} \int_{\xi_1^b}^{\xi_1} \mathrm{d}\hat{\xi}_1 \mathrm{e}^{\mathrm{i}\kappa\hat{\xi}_1} \tilde{\rho}(\hat{\xi}_1, l, p)\right] \tag{72}$$

where

$$\kappa = 2\pi\sqrt{l^2 + p^2} \tag{73}$$

and ξ_1^a and ξ_2^a are arbitrary real constants. The inverse Fourier transform (70) of $\tilde{\Phi}^+$ gives Φ^+.

We want to give an example of this solution for a particular value of r^2. But Φ^+ cannot be computed explicitly for the particular case given by (66). Let us consider

$$r^2 = aH(\xi')\delta(\eta) \tag{74}$$

which corresponds to the top front of a long pulse of a wave that has been emitted through a long and narrow slot. Then

$$\Phi^+ = \frac{-aPv^2}{2d^2}\left[\delta\left(\eta + \frac{\xi'}{d}\right) + \delta\left(\eta - \frac{\xi'}{d}\right)\right]. \tag{75}$$

The emitted wave Φ^+ is concentrated on the two half-planes $\varepsilon = \pm\xi'/d$, instead of being finite everywhere and regular, as in the case $V_0^2 > v^2$. The slowly varying waves emitted at various instants arrive all at the same time on these half-planes, and thus a very high peak is created. This effect is analogous to the formation of the boom of an airplane flying at a supersonic speed, in the case of sound waves. Although the explicit computation is not possible, we think that, in the case where the top front of the wave is punctual, the emitted wave will concentrate on a cone, as in the case of a supersonic boom.

Let us now have a look to the particular solution obtained when there is no transversal modulation of the incident wave, that is, when r depends only on ξ'. Then equation (58) has the very simple solution:

$$\Phi^+ = \frac{Pv^2}{V_0^2 - v^2} r^2. \tag{76}$$

We have

$$\frac{Pv^2}{V_0^2 - v^2} = -J_\xi. \tag{77}$$

Thus, using equation (145), we obtain

$$\Psi_1^+ \equiv 0. \tag{78}$$

If we assume that there is no incident solitary wave, we obtain finally

$$\boldsymbol{H}_2^0 = -\boldsymbol{M}_2^0 = -J_\xi r^2(\xi')\begin{pmatrix}1\\0\\0\end{pmatrix}. \tag{79}$$

This particular case had to be mentioned owing to its remarkable simplicity.

5. Effects on the phase of the rapidly oscillating wave

The solitary waves supported by $\boldsymbol{H}_2^0 = (\Phi, \Psi, \Xi)$ react on the phase θ of the rapidly oscillating wave, through the term $(C/A)\int_{\tau_1}^{\tau'}\Phi(\boldsymbol{\xi}', \hat{\tau})\,\mathrm{d}\hat{\tau}$ of equation (57) in the longitudinal case, and also through the term $(D/A)\int_{\tau_2}^{\tau'}\Psi(\boldsymbol{\xi}', \hat{\tau})\,\mathrm{d}\hat{\tau}$ of equation (25) in the general case. Φ and Ψ can always be written as the sum of two terms

$$\Phi = \Phi^0 + \Phi^+ \qquad \Psi = \Psi^0 + \Psi^+ \tag{80}$$

where Φ^0 and Ψ^0 are 'free' long solitary waves, and Φ^+, Ψ^+ constant in the frame moving at the group velocity of the rapidly oscillating wave. The phase θ reads thus:

$$\theta = \left[\Lambda r^2(\boldsymbol{\xi}') + \frac{C}{A}\Phi^+(\boldsymbol{\xi}') + \frac{D}{A}\Psi^+(\boldsymbol{\xi}')\right]\tau' + \frac{C}{A}\int_{\tau_1}^{\tau'}\Phi^0(\boldsymbol{\xi}', \hat{\tau})\,\mathrm{d}\hat{\tau} + \frac{D}{A}\int_{\tau_2}^{\tau'}\Psi^0(\boldsymbol{\xi}', \hat{\tau})\,\mathrm{d}\hat{\tau} + \theta_0(\boldsymbol{\xi}') \tag{81}$$

($\theta_0(\boldsymbol{\xi}')$ is an integration constant). In the absence of incident solitary waves, θ is thus proportional to the time τ', in the frame that moves at the group velocity. We restrict ourselves now to the longitudinal case, and assume that $\Phi^0 = 0$. Let us first consider the case of a plane wave, that is, the particular case, mentioned at the end of the previous section, where the envelope amplitude r varies only as a function of the single variable $\xi' = \xi - v\tau$. Then the phase θ is easy to compute. Equation (59) has the very simple solution (76). Thus, if no incident solitary wave does exist, the phase θ is obtained as

$$\theta = \Lambda_1 r^2 \tau' \tag{82}$$

with

$$\Lambda_1 = \frac{-2m\nu^3(1+\alpha+\delta\nu)}{(\alpha+\delta\nu)^4[2(\alpha+\delta\nu)(\alpha+\delta\nu+1)-\delta\nu]}. \tag{83}$$

(The notation is that of appendix 3.) In this particular case, the nonlinear effect is simply a phase factor proportional to the time and the intensity of the wave. As already stated, it has already been described in the NLS approximation [3]; the coefficient Λ_1 has the same value as the ratio D_0/A_0 in equation (34).

For an amplitude modulation depending on the transverse coordinates (η, ζ), we have, still in the longitudinal case, and in the absence of incident solitary waves:

$$\theta = \left[\Lambda r^2(\boldsymbol{\xi}') + \frac{C}{A}\Phi^+(\boldsymbol{\xi}')\right]\tau' + \theta_0(\boldsymbol{\xi}') \tag{84}$$

where Φ^+ is one of the two solutions given by equation (65) or (70), (72). Let us consider the particular examples studied in the previous section. In the case where $V_0 > v$, we consider the particular amplitude given by equation (66). Then,

$$\theta = \Lambda r^2(\boldsymbol{\xi}')\tau' + \Theta_1\frac{\xi'\tau'}{[\xi'^2 + d^2(\eta^2+\zeta^2)]^{3/2}} + \theta_0(\boldsymbol{\xi}') \tag{85}$$

with

$$\Theta_1 = \frac{CaPv^2}{A}. \tag{86}$$

Recall that $r^2(\boldsymbol{\xi}')$ has here the expression (66), and that

$$d = \sqrt{|V_0^2 - v^2|}. \tag{87}$$

The part of the phase θ that comes from the interaction is smooth. In the case where the fast oscillating wave travels faster than a solitary wave ($V_0 > v$), we consider the particular amplitude given by equation (74). Then

$$\theta = \Lambda r^2(\boldsymbol{\xi}')\tau' + \Theta_2 \left[\delta\left(\eta + \frac{\xi'}{d}\right) + \delta\left(\eta - \frac{\xi'}{d}\right)\right]\tau' + \theta_0(\boldsymbol{\xi}') \tag{88}$$

with

$$\Theta_2 = \frac{-CaPv^2}{2Ad^2}. \tag{89}$$

The term that comes from the interaction is now singular. It behaves as a supersonic boom in the case of sound waves. One can expect that, for more realistic values of the amplitude $r(\boldsymbol{\xi}')$, this singularity should still exist.

The next question is whether this effect will be observable. The first objection is that outside the support of r, which is a straight line in the space (or a plane in the second case), θ has no meaning, and so does expression (85). But we can easily avoid this drawback, by adding to the quantity r^2 of equation (66) a term that does not depend on ξ', and that does not vanish at the point where we intend to measure the phase θ. This will physically represent a rapid and localized increase of a wave, that is being emitted with a constant amplitude for a long time. Then, because of the operator ∂_ξ^2 in equation (64), and of the linearity of the equations, the expression (85), or (88) of θ is still valid.

Then, we may have two behaviours for the phase of the wave, depending whether $V_0 > v$ or $v > V_0$. If the wave is polarized with a negative helicity, these two cases are obtained depending on whether $\omega > \omega_0$ or not, where ω_0 is a known function of $\alpha = \|\boldsymbol{H}_0^0\|/\|\boldsymbol{M}_0^0\|$, increasing (for large α at least) (see equations (173) and (60)). Thus, increasing the external field for a given frequency, we would observe a transition between a singular low-field regime ($\omega_0 < \omega$), and a smooth high-field regime ($\omega_0 > \omega$). This transition should appear if we are able to measure the difference between the phases of the wave at two different points of the same wave front (defined by the equality of the linear part of the phases). We hope that it will be possible by use of interference techniques.

6. Conclusion

In a way analogous to the mathematical theory of nonlinear geometric optics, the propagation of an electromagnetic wave with a high intensity in a ferromagnetic medium is described by a nonlinear transport equation. This happens for an intensity scale much larger than the scale where the formation of NLS solitons occurs, for the same space scale. But it may also happen for an unchanged intensity scale, if the space scale under consideration is much larger. In the simplest case, which is the case of a plane wave propagating along the direction of the exterior field, and modulated only in this direction, it gives rise to a phase modulation proportional to both the time and the square of the amplitude of the wave. This effect has already been described by other models.

More interesting is the fact that the rapidly oscillating wave generates slowly varying waves. These waves belong to already known propagation modes, that can, under certain circumstances, support KdV or mKdV solitons. Two regimes occur, depending on whether the slowly varying waves travel slower or faster than the modulation of the rapidly oscillating wave. An analogue to the boom of a supersonic airplane can thus be created in these propagation modes. Furthermore, the slowly varying waves react on the phase of the fast oscillating wave. This is at the origin of a part of the phase modulation mentioned above. In the general case, the phase of the fast oscillating wave reflects quite precisely the amplitude of the slowly varying waves. In particular, the boom in the slowly varying wave can be observed in the phase modulation of the rapidly oscillating wave.

Appendix 1. Derivation of the transport equation

In this appendix, we give the details of the order-by-order resolution of the perturbative scheme of section 2. Equations (3), (4) read, after the expansion in a Fourier series:

$$(-n^2\omega^2 - 2\mathrm{i}n\omega\partial_t + \partial_t^2)(H^{n,x} + M^{n,x}) = -(\mathrm{i}nk + \partial_x)(\partial_y H^{n,y} + \partial_z H^{n,z}) + (\partial_y^2 + \partial_z^2)H^{n,x} \tag{90}$$

$$\begin{aligned}(-n^2\omega^2 - 2\mathrm{i}n\omega\partial_t + \partial_t^2)(H^{n,y} + M^{n,y}) = {}& -\partial_y((\mathrm{i}nk + \partial_x)H^{n,x} + \partial_z H^{n,z}) \\ & +(-n^2k^2 + 2\mathrm{i}nk\partial_x + \partial_x^2 + \partial_z^2)H^{n,y}\end{aligned} \tag{91}$$

$$\begin{aligned}(-n^2\omega^2 - 2\mathrm{i}n\omega\partial_t + \partial_t^2)(H^{n,z} + M^{n,z}) = {}& -\partial_z((\mathrm{i}nk + \partial_x)H^{n,x} + \partial_y H^{n,y}) \\ & +(-n^2k^2 + 2\mathrm{i}nk\partial_x + \partial_x^2 + \partial_y^2)H^{n,z}\end{aligned} \tag{92}$$

$$(-\mathrm{i}n\omega + \partial_t)\boldsymbol{M}^n = -\sum_{p+q=n} \boldsymbol{M}^p \wedge \boldsymbol{H}^q. \tag{93}$$

At order ε^0, it is found that $\boldsymbol{H}_0^0$ and $\boldsymbol{M}_0^0$ must be collinear. We obtain equations (10). Recall that we assume that α and $\boldsymbol{m}$ are constant, and that $\boldsymbol{H}_0^n = \mathbf{0}$ and $\boldsymbol{M}_0^n = \mathbf{0}$ for every non-zero n.

At order ε^1, we find the 'linear' solution (12), (13) with

$$\boldsymbol{h}_1^1 = \begin{pmatrix} \mathrm{i}\gamma\mu m_t \\ -\mathrm{i}\mu m_x \\ \gamma\omega \end{pmatrix} \tag{94}$$

$$\boldsymbol{m}_1^1 = \begin{pmatrix} -\mathrm{i}\gamma\mu m_t \\ \mathrm{i}\gamma\mu m_x \\ -\gamma^2\omega \end{pmatrix}. \tag{95}$$

(γ and ω are defined by equation (15).)

We impose $\boldsymbol{M}_1^n = \mathbf{0}$, $\boldsymbol{H}_1^n = \mathbf{0}$, if $|n| \neq 1$. In particular, $\boldsymbol{M}_1^0 = \mathbf{0}$, $\boldsymbol{H}_1^0 = \mathbf{0}$. This term would represent an incident solitary wave with a large space scale, and an amplitude of the same order of magnitude as the rapidly oscillating wave. We assume that there is no such wave; we retain the possibility to investigate the interaction between the rapidly oscillating waves and long solitary waves, but with a much smaller amplitude.

At order ε^2, we find that all terms $\boldsymbol{M}_2^n$, $\boldsymbol{H}_2^n$, are zero for $|n| \geqslant 3$, and that $\boldsymbol{M}_2^2$ and $\boldsymbol{H}_2^2$ are non-zero and defined in a unique way. They can be computed explicitly, but their expressions are not useful for our purposes. The fundamental harmonic can be expressed in the same way as at order ε:

$$\boldsymbol{H}_2^1 = \boldsymbol{h}_1^1 f \qquad \boldsymbol{M}_2^1 = \boldsymbol{m}_1^1 f \tag{96}$$

where f is an arbitrary function of $(\boldsymbol{\xi}, \tau)$. Equation (93) gives for $n = 0$, at this order:

$$H_2^{0,z} = \alpha M_2^{0,z} \tag{97}$$

$$m_x(H_2^{0,y} - \alpha M_2^{0,y}) - m_t(H_2^{0,x} - \alpha M_2^{0,x}) = -2\gamma(1-\gamma)\mu^2 m_x m_t |g|^2. \quad (98)$$

As stated above, equations (90)–(92) are trivial at order ε^2, thus the equations that relate $\boldsymbol{M}_2^0$ to $\boldsymbol{H}_2^0$ must be sought at order ε^6. These are equations (17).

At order ε^3, for $n = 1$, equations (90)–(92) give

$$\boldsymbol{M}_3^1 = \begin{pmatrix} -H_3^{1,x} \\ -\gamma H_3^{1,y} \\ -\gamma H_3^{1,z} \end{pmatrix} + \boldsymbol{P} \quad (99)$$

with the vector $\boldsymbol{P}$ given by

$$P^x = \frac{k}{\omega^2}(\mu m_x \partial_\eta + \mathrm{i}\gamma\omega\partial_\zeta) g \quad (100)$$

$$P^y = \frac{-\mu}{\omega^2}[2(1-\gamma)\omega m_x \partial_\tau + k(2m_x\partial_\xi + \gamma m_t \partial_\eta)] g \quad (101)$$

$$P^z = \frac{-\mathrm{i}\gamma}{\omega^2}[2(1-\gamma)\omega^2\partial_\tau + k(2\omega\partial_\xi - \mathrm{i}\mu m_t \partial_\zeta)] g. \quad (102)$$

Equation (93) gives, at this order,

$$-\mathrm{i}\omega \boldsymbol{M}_3^1 + \boldsymbol{m} \wedge (\boldsymbol{H}_3^1 - \alpha \boldsymbol{M}_3^1) = -\boldsymbol{S} \quad (103)$$

with

$$\boldsymbol{S} = \sum_{p+q=1} (\boldsymbol{M}_1^p \wedge \boldsymbol{H}_2^q + \boldsymbol{M}_2^p \wedge \boldsymbol{H}_1^q) + \partial_\tau \boldsymbol{M}_1^1. \quad (104)$$

After computation,

$$\boldsymbol{S} = \boldsymbol{s}' g + \boldsymbol{s}'' g|g|^2 + \boldsymbol{m}_1^1 \partial_\tau g \quad (105)$$

with

$$\boldsymbol{s}' = \begin{pmatrix} \gamma\omega(M_2^{0,y} + \gamma H_2^{0,y}) + \frac{\mathrm{i}\mu^2 m_x}{\alpha} H_2^{0,z} \\ -\gamma\omega(M_2^{0,x} + \gamma H_2^{0,x}) + \frac{\mathrm{i}\gamma\mu m_t}{\alpha}(1+\alpha) H_2^{0,z} \\ -\mathrm{i}\mu m_x (M_2^{0,x} + \gamma H_2^{0,x}) - \mathrm{i}\gamma\mu m_t (M_2^{0,y} + H_2^{0,y}) \end{pmatrix} \quad (106)$$

$$\boldsymbol{s}'' = \frac{-(1-\gamma)^2 \mu^2 m_t^2}{2\omega^2} \begin{pmatrix} 0 \\ \gamma\omega\mu m_x \\ \mathrm{i}(\gamma^2\omega^2 + 2\mu^2 m_x^2) \end{pmatrix} \quad (107)$$

and $\boldsymbol{m}_1^1$ is given by equation (95). Equation (103) can be written:

$$\boldsymbol{L}\boldsymbol{H}_3^1 = \mathrm{i}\omega\boldsymbol{P} + \alpha\boldsymbol{m} \wedge \boldsymbol{P} - \boldsymbol{S} \quad (108)$$

with

$$\boldsymbol{L}\boldsymbol{H} = -\mathrm{i}\omega \begin{pmatrix} -H^x \\ -\gamma H^y \\ -\gamma H^z \end{pmatrix} + \boldsymbol{m} \wedge \begin{pmatrix} (1+\alpha)H^x \\ \mu H^y \\ \mu H^z \end{pmatrix} = \begin{pmatrix} \mathrm{i}\omega & 0 & \mu m_t \\ 0 & \mathrm{i}\gamma\omega & -\mu m_x \\ -(1+\alpha)m_t & \mu m_x & \mathrm{i}\gamma\omega \end{pmatrix} \boldsymbol{H}. \quad (109)$$

The solvability condition of equation (108) is

$$\det \left[\begin{pmatrix} \mathrm{i}\gamma\omega \\ 0 \\ -(1+\alpha)m_t \end{pmatrix}, \begin{pmatrix} 0 \\ \mathrm{i}\gamma\omega \\ \mu m_x \end{pmatrix}, \mathrm{i}\omega\boldsymbol{P} + \alpha\boldsymbol{m} \wedge \boldsymbol{P} - \boldsymbol{S} \right] = 0. \quad (110)$$

It is equation (16). The coefficients of this equation are given by the following formulae:

$$A = \frac{2}{\mu}[(1-\gamma)\mu^2 m_x^2 + [\gamma\mu + (1-\gamma)]\gamma^2\omega^2] \tag{111}$$

$$\mathcal{D} = \frac{2k}{\omega\mu}[(\mu^2 m_x^2 + \gamma^2\omega^2)\partial_\xi + \gamma\mu^2 m_x m_t \partial_\eta] \tag{112}$$

$$B_0 = \frac{-\gamma(1-\gamma)^2\mu^2 m_t^2}{2\omega}(\gamma^2\omega^2 + 3\mu^2 m_x^2) \tag{113}$$

$$F(\boldsymbol{H}_2^0, \boldsymbol{M}_2^0) = -\{\gamma^2\omega m_t[(1+\alpha+\mu)M_2^{0,y} + (\gamma(1+\alpha)+\mu)H_2^{0,y}] \\ +2\gamma\omega\mu m_x(M_2^{0,x} + \gamma H_2^{0,x})\}. \tag{114}$$

We now intend to eliminate the components of $\boldsymbol{M}_2^0$ from equation (16), in order to describe the corresponding solitary waves conveniently.

First we compute the solution $\boldsymbol{H}_3^1$ to system (108), and the corresponding magnetization $\boldsymbol{M}_3^1$. Then we write the solvability condition for equation (93), for $n = 0$, at order ε^4; it reads

$$\boldsymbol{m} \wedge (\boldsymbol{H}_4^0 - \alpha\boldsymbol{M}_4^0) = -\boldsymbol{Q} \tag{115}$$

with

$$\boldsymbol{Q} = \partial_\tau \boldsymbol{M}_2^0 + \sum_{p+q=0} (\boldsymbol{M}_1^p \wedge \boldsymbol{H}_3^q + \boldsymbol{M}_2^p \wedge \boldsymbol{H}_2^q + \boldsymbol{M}_3^p \wedge \boldsymbol{H}_1^q). \tag{116}$$

The solvability condition is:

$$\boldsymbol{m} \cdot \boldsymbol{Q} = 0. \tag{117}$$

Among various terms, the quantity $\boldsymbol{m} \cdot \boldsymbol{Q}$ contains a term proportional to $H_2^{0,z}|g|^2$ and the term

$$\boldsymbol{m} \cdot \sum_{p+q=0} (\boldsymbol{M}_2^p \wedge \boldsymbol{H}_2^q) = \frac{H_2^{0,z}}{\alpha}[-m_x(H_2^{0,y} - \alpha M_2^{0,y}) + m_t(H_2^{0,x} - \alpha M_2^{0,x})]. \tag{118}$$

Using equation (98), this expression reduces to a term proportional to $H_2^{0,z}|g|^2$ that cancels the previous one.

Then equation (117) reduces to

$$m_x\partial_\tau M_2^{0,x} + m_t\partial_\tau M_2^{0,y} = \gamma(1-\gamma)[4\mu m_x^2 + \gamma m_t^2(2(1+\alpha)-\mu)]\partial_\tau|g|^2 \\ +\frac{2k\gamma}{\omega}[2\mu m_x^2 + \gamma(1+\alpha)m_t^2]\partial_\xi|g|^2 + 2\gamma^2\frac{k\mu}{\omega}m_x m_t\partial_\eta|g|^2. \tag{119}$$

After integration, equations (119) and (98) yield a 2×2 system for $M_2^{0,x}$ and $M_2^{0,y}$, that can be solved in terms of g, $H_2^{0,x}$, $H_2^{0,y}$.

Using the notation (18) for $H_2^{0,s}(s = x, y)$, and the expressions for $M_2^{0,s}(s = x, y)$ just computed in equation (114), we compute $F(\boldsymbol{H}_2^0, \boldsymbol{M}_2^0)$:

$$F(\boldsymbol{H}_2^0, \boldsymbol{M}_2^0) = C\Phi + D\Psi + F_0|g|^2 + E\int_{-\infty}^{\tau}\mathcal{D}|g|^2. \tag{120}$$

We have:

$$C = \frac{-\gamma\omega m_x}{\alpha m^2}(2\alpha\gamma\mu m_x^2 + [2\mu^2 - \gamma(1+\alpha+\mu)]m_t^2) \tag{121}$$

$$D = \frac{-\gamma\omega m_t}{\alpha m^2}(\alpha\gamma[\gamma(1+\alpha)+\mu]m_t^2 + 2\mu[(1+\alpha)\gamma - 1]m_x^2) \tag{122}$$

$$F_0 = \frac{-\gamma^2(1-\gamma)\omega}{\alpha m^2}[\gamma(1+\alpha+\mu)m_t^2([4\alpha\mu+2\mu^2]m_x^2+\alpha\gamma[2(1+\alpha)-\mu]m_t^2) \\ +2\mu m_x^2(4\alpha\mu m_x^2+[(2(1+\alpha)-\mu)\alpha\gamma-2\mu^2]m_t^2)] \tag{123}$$

$$E = \frac{-\gamma^2\omega}{m^2}[\gamma(1+\alpha+\mu)m_t^2+2\mu m_x^2] \tag{124}$$

and $\mathcal{D}$ given by (112). We also use the expressions for $M_2^{0,x}$ and $M_2^{0,y}$ in order to reduce equations (17). Finally the system (19)–(22) is obtained. The constants involved in these equations are as follows: A is given by equation (111), $\mathcal{D}$ by equation (112), C by equation (121), D by equation (122), E by equation (124), and

$$B = B_0 + F_0. \tag{125}$$

B_0 is given by (113), F_0 by (123), and

$$a = \frac{\gamma(1-\gamma)m_x}{\alpha m^2}(4\alpha\mu m_x^2+[2(1+\alpha)\alpha\gamma-3\mu^2+\mu]m_t^2) \tag{126}$$

$$b = \frac{\gamma(1-\gamma)m_t}{\alpha m^2}([4\alpha\mu+2\mu^2]m_x^2+\alpha\gamma[2(1+\alpha)-\mu]m_t^2). \tag{127}$$

This completes the derivation of the interaction system.

Appendix 2. Reduction of the interaction system

In this appendix, the interaction system (19)–(22) is reduced to explicit expressions (equations (40), (41), etc) and the evolution equation (46). The function g is decomposed into amplitude and phase as in equation (23). r and θ are real functions to be determined. Equation (19) reduces to

$$A\partial_\tau r + \mathcal{D}r = 0 \tag{128}$$

$$A\partial_\tau\theta + \mathcal{D}\theta = Br^2 + C\Phi + D\Psi + E\int_{-\infty}^{\tau}\mathcal{D}r^2. \tag{129}$$

Equation (128) shows that the amplitude r of the wave propagates without deformation; and equation (129) can be integrated to give the expression of the phase θ of the rapidly oscillating wave: we see that it is affected by the square amplitude r^2 of the wave itself, but also by interaction with slowly varying solitary waves described by Φ and Ψ. Using the change of variables (26), equation (128) reduces to

$$\partial_{\tau'}r = 0 \tag{130}$$

that is equation (24). Using equation (130), equation (129) reduces to

$$\partial_{\tau'}\theta = \Lambda r^2 + \frac{C}{A}\Phi + \frac{D}{A}\Psi \tag{131}$$

with

$$\Lambda = \frac{B}{A} - E. \tag{132}$$

Thus we obtain equation (25). Second we have to reduce equations (20)–(22), that give the evolution of Φ and Ψ, in order for the structure of the waves described by (Φ, Ψ, Ξ) to become noticeable. Using equation (128), they can be written as

$$\left(\partial_\eta^2+\partial_\zeta^2-\frac{\alpha+\sin^2\varphi}{\alpha}\partial_\tau^2\right)\Phi = \frac{-\sin\varphi\cos\varphi}{\alpha}\partial_\tau^2\Psi + \partial_\xi(\partial_\eta\Psi+\partial_\zeta\Xi) + J_\xi\partial_\tau^2 r^2 \tag{133}$$

$$\left(\partial_\xi^2+\partial_\zeta^2-\frac{\alpha+\cos^2\varphi}{\alpha}\partial_\tau^2\right)\Psi = \frac{-\sin\varphi\cos\varphi}{\alpha}\partial_\tau^2\Phi + \partial_\eta(\partial_\xi\Phi+\partial_\zeta\Xi) + J_\eta\partial_\tau^2 r^2 \tag{134}$$

with

$$J_\xi = a - \frac{A\gamma m_x}{m^2} \tag{135}$$

$$J_\eta = b - \frac{A\gamma m_t}{m^2}. \tag{136}$$

The following rotation enables us to reduce the system: the coordinates (X, Y) are defined by

$$\begin{aligned} \partial_X &= \cos\varphi\partial_\xi + \sin\varphi\partial_\eta \\ \partial_Y &= -\sin\varphi\partial_\xi + \cos\varphi\partial_\eta \end{aligned} \tag{137}$$

and the fields Φ_1 and Φ_2 by

$$\begin{aligned} \Phi &= \cos\varphi\Phi_1 - \sin\varphi\Phi_2 \\ \Psi &= \sin\varphi\Phi_1 + \cos\varphi\Phi_2. \end{aligned} \tag{138}$$

In this coordinate frame, the system becomes:

$$(\partial_Y^2 + \partial_\zeta^2 - \partial_\tau^2)\Phi_1 = \partial_X(\partial_Y\Phi_2 + \partial_\zeta\Xi) + J_X\partial_\tau^2 r^2 \tag{139}$$

$$\left(\partial_X^2 + \partial_\zeta^2 - \frac{\alpha+1}{\alpha}\partial_\tau^2\right)\Phi_2 = \partial_Y(\partial_X\Phi_1 + \partial_\zeta\Xi) + J_Y\partial_\tau^2 r^2 \tag{140}$$

$$\left(\partial_X^2 + \partial_Y^2 - \frac{\alpha+1}{\alpha}\partial_\tau^2\right)\Xi = \partial_\zeta(\partial_X\Phi_1 + \partial_Y\Phi_2). \tag{141}$$

The constants J_X, J_Y are defined by

$$\begin{aligned} J_X &= \cos\varphi J_\xi + \sin\varphi J_\eta \\ J_Y &= -\sin\varphi J_\xi + \cos\varphi J_\eta. \end{aligned} \tag{142}$$

The system (139)–(141) is decoupled by use of the transform (35) in the following way: deriving equation (140) with respect to ζ, and equation (141) with respect to Y, and subtracting, we get the evolution equation for Ψ_2:

$$\left(\partial_X^2 + \partial_Y^2 + \partial_\zeta^2 - \frac{1+\alpha}{\alpha}\partial_\tau^2\right)\Psi_2 = J_Y\partial_\zeta\partial_\tau^2 r^2. \tag{143}$$

Then, deriving equation (140) with respect to Y and equation (141) with respect to ζ, and adding up, we get

$$\left(\partial_X^2 - \frac{1+\alpha}{\alpha}\partial_\tau^2\right)\Psi_1 = (\partial_Y^2 + \partial_\zeta^2)\partial_X\Phi_1 + J_Y\partial_Y\partial_\tau^2 r^2. \tag{144}$$

We derive equation (139) with respect to X, and then add to equation (144), to obtain, after integration:

$$\Psi_1 = \frac{-\alpha}{1+\alpha}[\partial_X\Phi_1 + (J_X\partial_X + J_Y\partial_Y)r^2]. \tag{145}$$

Using this relation in equation (144), we get the evolution equation for Φ_1:

$$\left(\frac{\alpha}{1+\alpha}\partial_X^2 + \partial_Y^2 + \partial_\zeta^2 - \partial_\tau^2\right)\Phi_1 = \left(\frac{-\alpha}{1+\alpha}\partial_X[J_X\partial_X + J_Y\partial_Y] + J_X\partial_\tau^2\right)r^2. \tag{146}$$

The system (133), (134), (22), by means of the rotation of the coordinate axes (137), (138), and of the differential transform (35), has been reduced to two differential equations, one for Φ_1 only (equation (146)), and one for Ψ_2 only (equation (143)), and an explicit expression for Ψ_1 (equation (145)).

These equations are linear in Φ_1, Ψ_1, Ψ_2, thus is it worth considering the homogeneous equations associated with them (obtained by setting $r = 0$ in equations (143), (145), (146)). They read:

$$(V_0^2\partial_X^2 + \partial_Y^2 + \partial_\zeta^2 - \partial_\tau^2)\Phi_1^0 = 0 \tag{147}$$

$$\Psi_1^0 = -V_0^2\partial_X\Phi_1^0 \tag{148}$$

$$(V_0^2(\partial_X^2 + \partial_Y^2 + \partial_\zeta^2) - \partial_\tau^2)\Psi_2^0 = 0. \tag{149}$$

We call Φ_1^0, Ψ_1^0, Ψ_2^0 the part functions Φ_1, Ψ_1, Ψ_2 which is the solution of these homogeneous equations, and V_0 is given by equation (39). Equations (147)–(149) enable us to give expressions for $\boldsymbol{H}_2^0$ and $\boldsymbol{M}_2^0$, in the following way: the functions χ_j, $j = 1, 2$, are defined by

$$(\partial_Y^2 + \partial_\zeta^2)\chi_1 = \Phi_1^0 \tag{150}$$

$$(\partial_Y^2 + \partial_\zeta^2)\chi_2 = \Psi_2^0 \tag{151}$$

and the χ_j vanish at infinity. The evolution equations for χ_1 and χ_2 are the same as for Φ_1^0 and Ψ_2^0 (equations (37) and (38)). Then equations (36) can be integrated to give expressions (40) and (41) of $\boldsymbol{H}_2^0$ and $\boldsymbol{M}_2^0$ as functions of χ_1 and χ_2.

Now we seek a particular solution of the complete equations. As written in section 3, we can search a particular solution Φ^+ of equation (146), function of $\boldsymbol{\xi}'$ only. If Φ_1^+ is so, the general solution of equation (146) is

$$\Phi_1(\boldsymbol{\xi}', \tau') = \Phi_1^+(\boldsymbol{\xi}') + \Phi_1^0(\boldsymbol{\xi}', \tau') \tag{152}$$

where Φ_1^0 is a solution of equation (147), that is, an incident solitary wave of the above-mentioned type. As Φ_1^+ is a function of $\boldsymbol{\xi}' = (\xi - v_\xi\tau, \eta - v_\eta\tau, \zeta)$,

$$\partial_\tau\Phi_1^+ = -(v_\xi\partial_\xi + v_\eta\partial_\eta)\Phi_1^+. \tag{153}$$

In the rotated frame $(XY\zeta)$, we have:

$$v_\xi\partial_\xi + v_\eta\partial_\eta = v_X\partial_X + v_Y\partial_Y \tag{154}$$

with

$$\begin{aligned} v_X &= \cos\varphi v_\xi + \sin\varphi v_\eta \\ v_Y &= -\sin\varphi v_\xi + \cos\varphi v_\eta. \end{aligned} \tag{155}$$

Computation gives

$$v_X = \frac{v_\xi m_x}{m(b+1)(1+\alpha)}[1 + \alpha + \mu + b\alpha(1-\gamma)] \tag{156}$$

$$v_Y = \frac{-v_\xi m_t}{m(b+1)}[1 + b(1-\gamma)]. \tag{157}$$

Thus equation (146) becomes equation (46) for Φ_1^+. $\mathcal{R}$ given by

$$\mathcal{R} = [J_X(v_X^2 - V_0^2)\partial_X^2 + (2J_X v_X v_Y - J_Y V_0^2)\partial_X\partial_Y + J_X v_Y^2\partial_Y^2]r^2. \tag{158}$$

Appendix 3. An explicit computation in the longitudinal case

In this appendix, we study the special case where the exterior field is parallel to the propagation direction. We give some details of the calculus, and the explicit expression of the coefficients. The dispersion relation (14) can be solved, and we obtain

$$\gamma = \frac{-1}{\alpha + \delta\nu} \tag{159}$$

with $\nu = \omega/m$ and $\delta = \pm 1$. $\delta = 1(-1)$ corresponds to a wave with positive (negative) helicity. Thus the computation can be achieved explicitly. The evolution of the waves is governed by the system (51)–(54), where the coefficients are given by

$$A = \frac{2m^2\delta\nu}{(\alpha+\delta\nu)^3}[2(\alpha+\delta\nu)(\alpha+\delta\nu+1)-\delta\nu] \tag{160}$$

$$B = \frac{-8m^3\nu^3(\alpha+\delta\nu+1)}{(\alpha+\delta\nu)^5} \tag{161}$$

$$C = mE = \frac{-2m^2\delta\nu^2}{(\alpha+\delta\nu)^3} \tag{162}$$

$$K = \frac{4m^2\delta\nu(\alpha+\delta\nu+1)^{1/2}}{(\alpha+\delta\nu)^{3/2}} \tag{163}$$

$$a = \frac{-4m\delta\nu(\alpha+\delta\nu+1)}{(\alpha+\delta\nu)^3}. \tag{164}$$

The reduction of the system is analogous to the general case. We obtain first equations (24) and (57), where the constants Λ and C/A are given by

$$\Lambda = \frac{-2m\nu^3}{(\alpha+\delta\nu)^3[2(\alpha+\delta\nu)(\alpha+\delta\nu+1)-\delta\nu]} \tag{165}$$

$$\frac{C}{A} = \frac{-\nu}{2(\alpha+\delta\nu)(\alpha+\delta\nu+1)-\delta\nu}. \tag{166}$$

Equations (20), (21) may be written in a much simpler way as in the previous section, and the rotation (137) is trivial. The coefficient $J_Y = J_\eta$ is zero. The transform (35) reads:

$$\begin{aligned}\Psi_1 &= \partial_\eta\Psi + \partial_\zeta\Xi \\ \Psi_2 &= \partial_\zeta\Psi - \partial_\eta\Xi.\end{aligned} \tag{167}$$

Equation (146), which gives Ψ_1 in terms of $\Phi_1 = \Phi$, is still valid with

$$J_Y = J_\eta = 0 \tag{168}$$

and

$$J_X = J_\xi = \frac{-2m\nu^2}{(\alpha+\delta\nu)^4}. \tag{169}$$

Equation (143) for Ψ_2 becomes the equation for a free wave propagation at velocity V_0. Then we obtain equation (58), with P given by

$$P = \frac{-m\delta\nu^3[4(\alpha+\delta\nu)(\alpha+\delta\nu+1)(2\alpha+\delta\nu)-\alpha\delta\nu]}{2(1+\alpha)(\alpha+\delta\nu)^7(\alpha+\delta\nu+1)}. \tag{170}$$

A particular solution Φ^+ of equation (58) verifies equation (59); it depends on the sign of the quantity $(V_0^2 - v^2)$. We have

$$V_0^2 - v^2 = \frac{-\delta\nu N}{Q} \tag{171}$$

with

$$Q = (1+\alpha)[2w^2 + w + \alpha]^2 \tag{172}$$

and

$$N = 4w^3 + 4(1+\alpha)w^2 + 3\alpha w + \alpha^2. \tag{173}$$

We use the parameter $w = \alpha + \delta\nu$, that takes every real value. Study of the function $N(w)$ shows that, for every $\alpha > 0$, N changes its sign only once, for a negative value $w_0 = \alpha - \nu_0$. The sign of $(V_0^2 - v^2)$ is deduced from this result; and discussed in section 4 of the paper. The threshold frequency ω_0 is defined by

$$\omega_0 = m\nu_0 = m(\alpha - w_0) \tag{174}$$

where w_0 is the unique real solution of the equation $N(w) = 0$.

References

[1] Bass F G and Nasonov N N 1990 Nonlinear electromagnetic-spin waves *Phys. Rep.* **189** 165

[2] Leblond H and Manna M 1994 Focusing and defocusing of electromagnetic waves in a ferromagnet *J. Phys. A: Math. Gen.* **27** 3245

[3] Leblond H and Manna M 1994 Benjamin–Feir type instability in a saturated ferrite. Transition between a focusing and defocusing regimen for polarized electromagnetic wave *Phys. Rev.* E **50** 2275

[4] De Gasperis P, Marcelli R and Miccoli G 1987 Magnetostatic soliton propagation at microwave frequency in magnetic garnet films *Phys. Rev. Lett.* **59** 481

[5] Kalinikos B A, Kovshikov N G and Slavin A N 1990 Experimental observation of magnetostatic wave envelope solitons in yttrium iron garnet films *Phys. Rev.* B **42** 8658

[6] Ming Chen, Tsankov M A, Nash J M and Patton C E 1993 Microwave magnetic-envelope dark-solitons in yttrium iron garnet thin films *Phys. Rev. Lett.* **70** 1707

[7] Taniuti T and Nishihara K 1983 *Nonlinear waves* (London: Pitman)

[8] Dodd R K, Eilbeck J C, Gibbon J D and Morris H C 1982 *Solitons and nonlinear wave equations* (London: Academic)

[9] Hunter J and Keller J 1983 Weakly nonlinear high frequency waves *Commun. Pure Appl. Math.* **36** 547

[10] Joly J L, Métivier G and Rauch J 1993 Resonant one-dimensional nonlinear geometric optics *J. Functional Analysis* **114** 106

[11] Donnat Ph 1994 *Quelques contributions mathématiques en optique non linéaire*, thesis of doctor of the Ecole Polytechnique, Paris

[12] Soohoo R F 1960 *Theory and application of ferrites* (London: Prentice-Hall)

[13] Walker A D M and McKenzie J F 1985 Properties of electromagnetic waves in ferrites *Proc. R. Soc.* A **399** 217

[14] Polder D 1949 On the theory of ferromagnetic resonance *Phil. Mag.* **40** 99

[15] Kalinikos B A and Slavin A N 1986 Theory of dipole-exchange spin wave spectrum for ferromagnetic films with mixed exchange boundary conditions *J. Phys. C: Solid State Phys.* **19** 7013

[16] Leblond H Electromagnetic waves in ferrites: from linear absorption to nonlinear Schrödinger equation, in preparation

[17] Suhl H 1957 The theory of ferromagnetic resonance at high signal powers *J. Phys. Chem. Solids* **1** 209

[18] Leblond H 1995 Nonlinear anomalous ferromagnetic Faraday effect *Phys. Rev.* E **52** 1081

[19] Nakata I 1991 Nonlinear electromagnetic waves in a ferromagnet *J. Phys. Soc. Japan* **60** 77

[20] Leblond H 1995 Interaction of two solitary waves in a ferromagnet *J. Phys. A: Math. Gen.* **28** 3763